Elementary results about prime numbers

Note: "Elementary" doesn't mean easy! (on the contrary!) It means no advanced techniques are used.

Recall: $\psi(x) = \sum_{n \le x} \Lambda(n)$

$\theta(x) = \sum_{p \le x} \log p$, $\quad \psi(x) = \theta(x) + O(\sqrt{x}\log x)$

TFAE: $\psi(x) \sim x$, $\theta(x) \sim x$, $\pi(x) \sim \frac{x}{\log x}$

Theorem (Chebyshev) We have that

$$\pi(x) \asymp \frac{x}{\log x}, \quad \theta(x) \asymp x, \quad \psi(x) \asymp x.$$

($\exists$ constants $C_1, C_2 > 0$ such that

$$C_1 \frac{x}{\log x} \le \pi(x) \le C_2 \frac{x}{\log x}, \quad C_1 x \le \theta(x) \le C_2 x, \quad C_1 x \le \psi(x) \le C_2 x \text{)}$$

Proof: Let $S(x) := \sum_{n \le x} \log n - 2 \sum_{n \le \frac{x}{2}} \log n$.

Recall $\sum_{n \le x} \log n = x \log x - x + O(\log x)$

(since log increasing)

$$\Longrightarrow S(x) = (\log 2)\cdot x + O(\log x). \quad (*)$$

Now we use $\log = \Lambda * \mathcal{E}$, so

$$\sum_{n\le x} \log n = \sum_{d\le x} \Lambda(d) \sum_{n \le \frac{x}{d}} 1 = \sum_{d\le x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor$$

$$\Rightarrow S(x) = \sum_{d\le x} \Lambda(d)\left\lfloor \frac{x}{d}\right\rfloor - 2\sum_{d\le \frac{x}{2}} \Lambda(d) \left\lfloor \frac{x}{2d} \right\rfloor$$

$$= \sum_{d\le x} \Lambda(d)\left(\left\lfloor \frac{x}{d}\right\rfloor - 2\left\lfloor \frac{x}{2d}\right\rfloor\right) \quad (**)$$

(we used that $\left\lfloor \frac{x}{2d} \right\rfloor = 0$ if $d > \frac{x}{2}$).

<u>Note</u>: For all $\alpha \ge 1$, $\lfloor \alpha \rfloor - 2\left\lfloor \frac{\alpha}{2}\right\rfloor \in \{0,1\}$

Also, for $1 \le \alpha < 2$, $\lfloor \alpha \rfloor - 2\left\lfloor \frac{\alpha}{2}\right\rfloor = 1$.

(exercise)

From $(**)$, this implies $\psi(x) - \psi\left(\frac{x}{2}\right) \le S(x) \le \psi(x)$ $\quad (***)$

since $\left\lfloor \frac{x}{d}\right\rfloor - 2\left\lfloor \frac{x}{2d}\right\rfloor = 1$ for $\frac{x}{2} < d \le x$

Hence, $\psi(x) \ge S(x) \gg x$ (from $(*)$)

On the other hand $\psi(x) - \psi\left(\frac{x}{2}\right) = O(x)$

$$\psi(x) = \psi(x) - \psi\left(\frac{x}{2}\right) + \psi\left(\frac{x}{2}\right) - \psi\left(\frac{x}{4}\right) + \dots + \psi\left(\frac{x}{2^j}\right) - \underbrace{\psi\left(\frac{x}{2^{j+1}}\right)}_{\text{eventually } 0}.$$

$$= \sum_{j=0}^{\infty} \left(\psi\left(\frac{x}{2^j}\right) - \psi\left(\frac{x}{2^{j+1}}\right) \right)$$

$$= \sum_{j=0}^{\infty} O\left(\frac{x}{2^j}\right) = O(x).$$

This proves $\psi(x) \asymp x$, and hence $\theta(x) \asymp x$.

(since $\psi(x) = \theta(x) + O(\sqrt{x} \log x)$).

To deduce $\pi(x) \asymp \frac{x}{\log x}$, first note

$$\pi(x) = \sum_{p \le x} 1 \ge \frac{1}{\log x} \sum_{p \le x} \log p = \frac{1}{\log x} \theta(x) \gg \frac{x}{\log x}$$

For the other direction,

$$\pi(x) \le \pi(\sqrt{x}) + \sum_{\sqrt{x} < p \le x} \frac{\log p}{\log \sqrt{x}}$$

$$\le \sqrt{x} + \frac{2\theta(x)}{\log x} \ll \frac{x}{\log x} \qquad \boxtimes$$

Mertens' theorems

We cannot count primes precisely (yet!), but we can obtain useful results if we add some "weights".

Theorem: $$\sum_{p\le x}\frac{\log p}{p}=\log x+O(1).$$

Proof: **Step 1**: $$\sum_{d\le x}\frac{\Lambda(d)}{d}=\log x+O(1).$$

Proof of claim: Note that since $\log=\Lambda*\varepsilon$,

$$\sum_{n\le x}\log n=\sum_{n\le x}\Lambda*\varepsilon(n)=\sum_{d\le x}\Lambda(d)\left\lfloor\frac{x}{d}\right\rfloor$$

$$=x\sum_{d\le x}\frac{\Lambda(d)}{d}+\underbrace{O(\psi(x))}_{=O(x)\text{ from Chebyshev}}$$

But $$\sum_{n\le x}\log n=x\log x+O(x).$$

$$\rightarrow x\sum_{d\le x}\frac{\Lambda(d)}{d}=x\log x+O(x)\ \checkmark$$

Step 2: $$\sum_{p\le x}\frac{\log p}{p}=\sum_{d\le x}\frac{\Lambda(d)}{d}+O(1).$$

$$\left|\sum_{p\le x}\frac{\log p}{p}-\sum_{d\le x}\frac{\Lambda(d)}{d}\right|=\sum_{p\le x^{1/2}}\log p\sum_{2\le m\le\log x/\log p}\frac{1}{p^k}$$

$$\leq \sum_{p \leq x^{1/2}} \frac{\log p}{p^2} \underbrace{\left(\frac{p}{p-1}\right)}_{\leq 2} \ll \sum_{p} \frac{\log p}{p^2}$$

$$\ll \sum_{n} \frac{\log n}{n^2} \ll 1.$$

exercise: ⊛

$$⊛ \quad \sum_{n} \frac{\log n}{n^2} \ll \sum_{n \geq 2} n^{-3/2} \ll \int_1^{\infty} t^{-3/2} dt = O(1)$$

<u>Theorem</u>: $\exists$ an absolute constant M such that, for $x \geq 3$, we have

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + M + O\left(\frac{1}{\log x}\right).$$

<u>Proof</u>: We deduce it from previous Mertens thm by partial summation.

Let $S(x) = \sum_{p \leq x} \frac{\log p}{p} = \log x + O(1)$

Hence $S(x) = \log x + E(x)$, where $E(x) \ll 1$.

$$\sum_{p \leq x} \frac{1}{p} = \frac{S(x)}{\log x} + \int_{3/2}^{x} \frac{S(t)}{(\log t)^2 \cdot t} dt$$

$$= \frac{\log x}{\log x} + \frac{E(x)}{x} + \int_{3/2}^{x} \frac{1}{\log t \cdot t}\,dt + \int_{3/2}^{x} \frac{E(t)}{(\log t)^2 t}\,dt$$

$$= \log\log x + \underbrace{\left(\int_{3/2}^{\infty} \frac{E(t)}{(\log t)^2 t}\,dt + 1 - \log\log\frac{3}{2} \right)}_{M}$$

$$+ O\left(\frac{1}{x} + \underbrace{\int_{x}^{\infty} \frac{1}{(\log t)^2 \cdot t}\,dt}_{= \frac{1}{\log x}} \right)$$

$$= \log\log x + M + O\left(\frac{1}{\log x}\right). \qquad \blacksquare$$

Theorem: $\exists A > 0$ constant such that for $x \geq 2$

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{A}{\log x} + O\left(\frac{1}{(\log x)^2}\right).$$

Proof: From Taylor expansion, for $y \in (-1, 1)$,

$$\log(1-y) = -\sum_{n=1}^{\infty} \frac{y^n}{n}.$$

Thus

$$\sum_{p \leq x} \log\left(1 - \frac{1}{p}\right) = -\sum_{p \leq x} \frac{1}{p} - \sum_{p \leq x} \sum_{n=2}^{\infty} \frac{1}{np^n}$$

$$= -\sum_{p \leq x} \frac{1}{p} - \underbrace{\sum_{p} \sum_{n=2}^{\infty} \frac{1}{np^n}}_{B} + O\left(\sum_{p > x} \sum_{n=2}^{\infty} \frac{1}{np^n} \right)$$

Note $$\sum_{p>x}\sum_{n=2}^{\infty}\frac{1}{np^n} \le \sum_{p>x}\frac{1}{p^2}\sum_{n=0}^{\infty}\frac{1}{p^n}$$

$$\le \sum_{p>x}\frac{1}{p(p-1)} \le \sum_{n>x}\frac{1}{n(n-1)} \ll \frac{1}{x}.$$

Therefore $$\sum_{p\le x}\log\left(1-\frac{1}{p}\right) = -\log\log x + (M+B) + O\left(\frac{1}{\log x}\right).$$

We use that for $|y|<1$, $\exp(y) = 1+O(y)$.

Hence $$\exp\left(O\left(\frac{1}{\log x}\right)\right) = 1 + O\left(\frac{1}{\log x}\right).$$

$$\Rightarrow \prod_{p\le x}\left(1-\frac{1}{p}\right) = \frac{e^{M+B}}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right) \qquad \square$$

<u>Fact</u>: It follows from the proof that
$$A = e^{-\gamma}.$$

Dirichlet series

Definition: Let $f \in \mathcal{A}$ and $s \in \mathbb{C}$. We define the Dirichlet series attached to f to be the formal series

$$L_f(s) := \sum_{n \in \mathbb{N}} \frac{f(n)}{n^s}.$$

Convention: Oftentimes in analytic number theory, if $s \in \mathbb{C}$, we write $s = \sigma + it$, where $\sigma = \mathrm{Re}(s)$ and $t = \mathrm{Im}(s)$.

Half planes of (absolute) convergence

Theorem: Let $f \in \mathcal{A}$. There exists $\sigma_a \in \mathbb{R} \cup \{\pm\infty\}$ such that $L_f(s)$ converges absolutely for for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma_a(f)$ and it does not converge absolutely for $\mathrm{Re}(s) < \sigma_a(f)$.

$\sigma_a(f)$ is called abscissa of absolute convergence.

Proof: Let $\Delta = \{ s \in \mathbb{C} : L_f(s)$ converges absolutely$\}$.
If $\Delta = \emptyset$, then set $\sigma_a = \infty$, nothing to prove.

We define $\sigma_a(f) := \inf \{ \operatorname{Re}(s) : s \in \Delta \} \in \mathbb{R} \cup \{-\infty\}$.

We need to show that if $\sigma = \operatorname{Re}(s) > \sigma_a(f)$, then $L_f(s)$ converges absolutely.

Let $\sigma > \sigma_a(f)$. Then there exists $\sigma_a(f) < \sigma' < \sigma$ and $s' = \sigma' + it'$ such that $L_f(s')$ absolutely convergent. (s' exists by definition)

Then
$$\sum_{n \in \mathbb{N}} \left| \frac{f(n)}{n^s} \right| = \sum_{n \in \mathbb{N}} \frac{|f(n)|}{n^\sigma} < \sum_{n \in \mathbb{N}} \frac{|f(n)|}{n^{\sigma'}} < \infty.$$

$\Rightarrow L_f(s)$ absolutely covergent. $\square$.

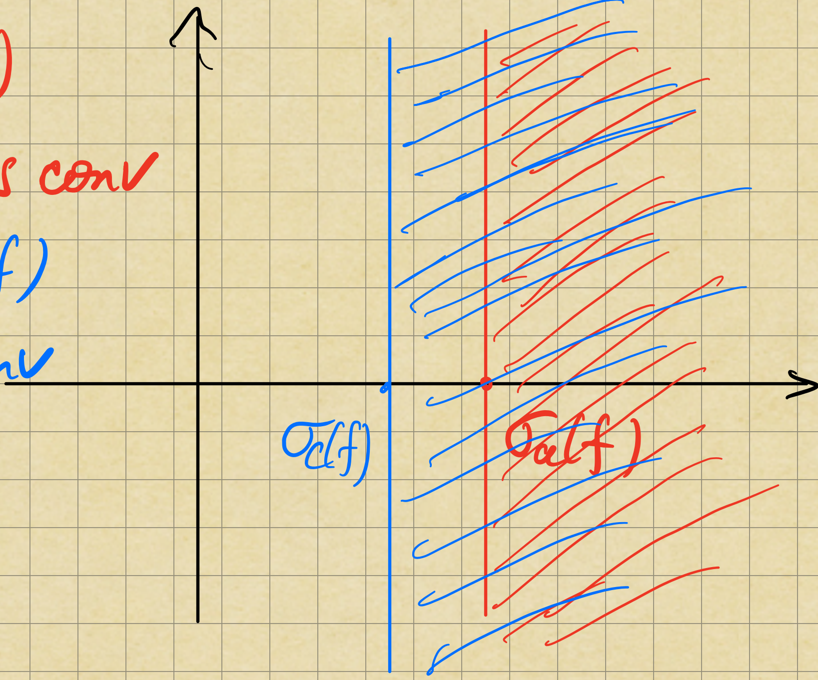
Theorem: Let $f \in \mathcal{A}$. There exists $\sigma_c(f) \in \mathbb{R} \cup \{\pm\infty\}$ (called abscissa of conditional convergence) such that $L_f(s)$ converges if $\sigma > \sigma_c(f)$ and $L_f(s)$ does not converge if $\sigma < \sigma_c(f)$.
The convergence is uniform on compact sets and $\sigma_a(f) - 1 \leq \sigma_c(f) \leq \sigma_a(f)$.

$Re(s) > \sigma_a(f)$

$\Rightarrow \mathcal{L}_f(s)$ abs conv

$Re(s) > \sigma_c(f)$

$\Rightarrow \mathcal{L}_f(s)$ conv



Proof next time!